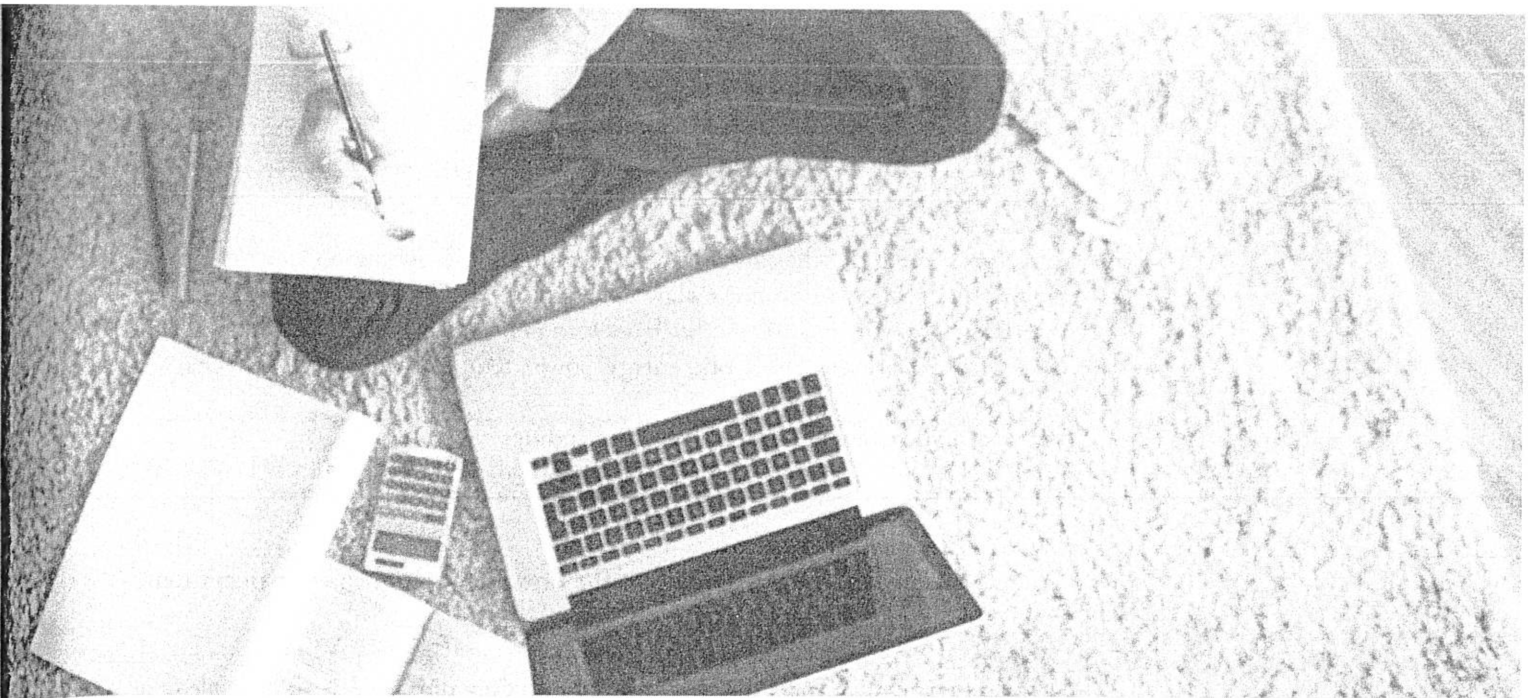


Chapter 3
Vectors

Vectors will show up all over the place in our study of physics. Some physical quantities that are represented as vectors are displacement, velocity, acceleration, force, momentum, and electric and magnetic fields. Since vectors play such a recurring role, it's important to become comfortable working with them; the purpose of this chapter is to provide you with a mastery of the fundamental vector algebra we'll use in subsequent chapters. For now, we'll restrict our study to two-dimensional vectors (that is, ones that lie flat in a plane).

DEFINITION

A **vector** is a quantity that involves both magnitude and direction. A quantity that does not involve direction is a **scalar**. For example, the quantity *55 miles per hour* is a scalar, while the quantity *55 miles per hour to the north* is a vector. Other examples of scalars include mass, work, energy, power, temperature, and electric charge.

Vectors can be denoted in several ways, including:

$$\mathbf{A},\ \boldsymbol{A},\ \bar{A},\ \vec{A}$$

In textbooks, you'll usually see one of the first two, but when it's handwritten, you'll see one of the last two.

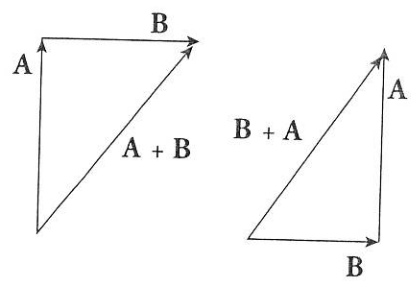
Displacement (which is net distance traveled including direction) is an example of a vector:

$\underbrace{\mathbf{A}}_{\text{displacement}} = \underbrace{4\text{ miles}}_{\text{magnitude}}\ \underbrace{\text{to the north}}_{\text{direction}}$ $\mathbf{B} = \underbrace{3\text{ miles}}_{\text{magnitude}}\ \underbrace{\text{to the east}}_{\text{direction}}$

Vectors obey the Commutative Law for Addition, which states:

Commutative Law for Addition

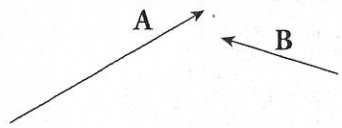
A + B = B + A



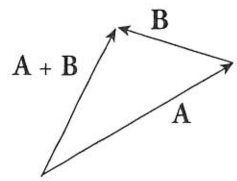
The vector sum **A** + **B** means the vector **A** followed by **B**. The vector sum of **B** + **A** means the vector **B** followed by **A**, and the result is an identical vector to **A** + **B**. Vectors are always added tail to end to find their sum, so **A** + **B** or **B** + **A**—both are examples of tail to end.

Two vectors are equal when they have the same magnitude and the same direction.

Example 1 Add the following two vectors:



Solution. Place the tail of **B** at the tip of **A** and connect them:



SCALAR MULTIPLICATION

A vector can be multiplied by a scalar (that is, by a number), and the result is a vector. If the original vector is A and the scalar is *k*, then the scalar multiple *k***A** is as follows:

Also Worth Noting!

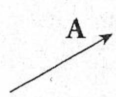
Scalar multiplication indicates a change in magnitude by the numerical multiple, and direction if there is a negative sign only (negative sign indicates opposite direction). An example of this is that of a car traveling east at a velocity of 100 km per hour. When multiplied by $k = -2$, the car's new velocity is 200 km per hour WEST.

Scalar Multiplication

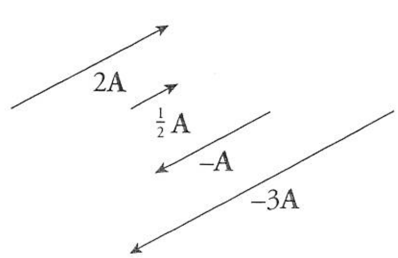
$$\text{magnitude of } k\mathbf{A} = |k| \times (\text{magnitude of } \mathbf{A})$$

$$\text{direction of } k\mathbf{A} = \begin{cases} \text{the same as } \mathbf{A} \text{ if } k \text{ is positive} \\ \text{the opposite of } \mathbf{A} \text{ if } k \text{ is negative} \end{cases}$$

Example 2 Sketch the scalar multiples 2**A**, $\frac{1}{2}$**A**, –**A**, and –3**A** of the vector **A**:



Solution.

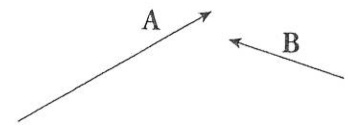


VECTOR SUBTRACTION

To subtract one vector from another, for example, to get A – B, simply form the vector –B, which is the scalar multiple (–1)B, and add it to A:

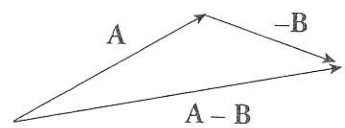
$$A - B = A + (-B)$$

Example 3 For the two vectors A and B, find the vector A – B.



Tail To End!
By adding the negative of B, we are allowing the process to follow the tail to end convention that we discussed earlier.

Solution. Flip B around (thereby forming –B) and add that vector to A:

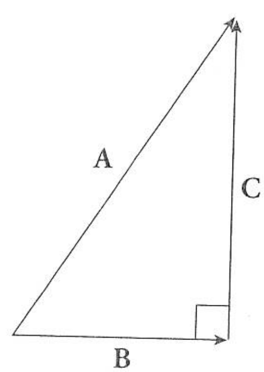


A Note About Direction
Make sure to pay attention to direction if you are not using a coordinate system. If you set a vector point to the right as positive, then you must set a vector pointing to the left as negative.

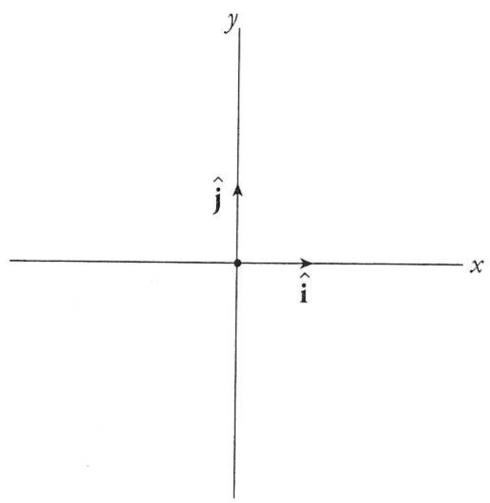
It is important to know that vector subtraction is **not** commutative; you must perform the subtraction in the order stated in the problem.

TWO-DIMENSIONAL VECTORS

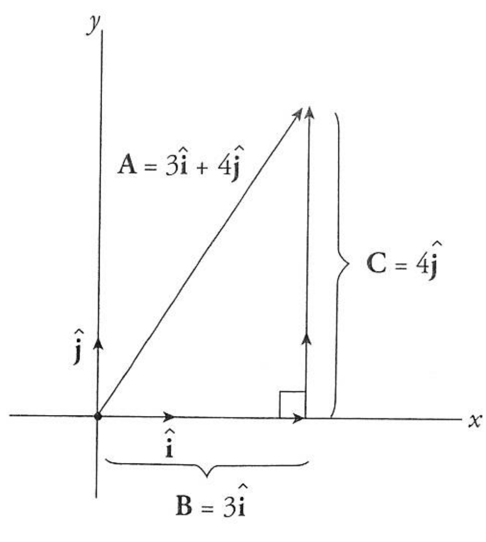
Two-dimensional vectors are vectors that lie flat in a plane and can be written as the sum of a horizontal vector and a vertical vector. For example, in the following diagram, the vector A is equal to the horizontal vector B plus the vertical vector C:



The horizontal vector is always considered a scalar multiple of what's called the **horizontal basis vector, i**, and the vertical vector is a scalar multiple of the **vertical basis vector, j**. Both of these special vectors have a magnitude of 1, and for this reason, they're called **unit vectors**. Unit vectors are often represented by placing a hat (caret) over the vector; for example, the unit **vectors i** and **j** are sometimes denoted î and ĵ.



For instance, the vector **A** in the figure below is the sum of the horizontal vector **B** = 3î and the vertical vector **C** = 4ĵ.



The vectors **B** and **C** are called the **vector components** of **A**, and the scalar multiples of î and ĵ which give **A**—in this case, 3 and 4—are called the **scalar components** of **A**. So vector **A** can be written as the sum $A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}}$, where A_x and A_y are the scalar components of **A**. The component A_x is called the **horizontal** scalar component of **A**, and A_y is called the **vertical** scalar component of **A**. In general, any vector in a plane can be described in this manner.

Coordinate System
Think of *i* as your *x*-coordinate system and *j* as your *y*-coordinate system. *i* is just a unit vector that points in the positive *x* direction, and *j* is just a unit vector that points in the positive *y* direction.

VECTOR OPERATIONS USING COMPONENTS

The use of components makes the vector operations of addition, subtraction, and scalar multiplication pretty straightforward:

Vector addition: *Add the respective components.*

$$\mathbf{A} + \mathbf{B} = (A_x + B_x)\mathbf{\hat{i}} + (A_y + B_y)\mathbf{\hat{j}}$$

Vector subtraction: *Subtract the respective components.*

$$\mathbf{A} - \mathbf{B} = (A_x - B_x)\mathbf{\hat{i}} + (A_y - B_y)\mathbf{\hat{j}}$$

Scalar multiplication: *Multiply each component by k.*

$$k\mathbf{A} = (kA_x)\mathbf{\hat{i}} + (kA_y)\mathbf{\hat{j}}$$

Example 4 If $\mathbf{A} = 2\mathbf{\hat{i}} - 3\mathbf{\hat{j}}$ and $\mathbf{B} = -4\mathbf{\hat{i}} + 2\mathbf{\hat{j}}$, compute each of the following vectors: **A** + **B**, **A** – **B**, 2**A**, and **A** + 3**B**.

Solution. It's very helpful that the given vectors **A** and **B** are written explicitly in terms of the standard basis vectors $\mathbf{\hat{i}}$ and $\mathbf{\hat{j}}$:

$$\mathbf{A} + \mathbf{B} = (2 - 4)\mathbf{\hat{i}} + (-3 + 2)\mathbf{\hat{j}} = -2\mathbf{\hat{i}} - \mathbf{\hat{j}}$$

$$\mathbf{A} - \mathbf{B} = [2 - (-4)]\mathbf{\hat{i}} + (-3 - 2)\mathbf{\hat{j}} = 6\mathbf{\hat{i}} - 5\mathbf{\hat{j}}$$

$$2\mathbf{A} = 2(2)\mathbf{\hat{i}} + 2(-3)\mathbf{\hat{j}} = 4\mathbf{\hat{i}} - 6\mathbf{\hat{j}}$$

$$\mathbf{A} + 3\mathbf{B} = [2 + 3(-4)]\mathbf{\hat{i}} + [-3 + 3(2)]\mathbf{\hat{j}} = -10\mathbf{\hat{i}} + 3\mathbf{\hat{j}}$$

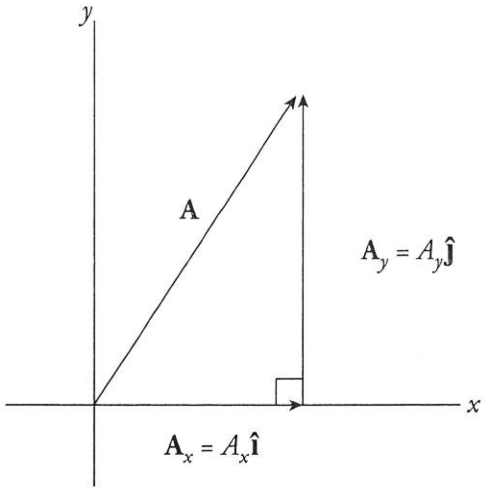
MAGNITUDE OF A VECTOR

This is merely an interpretation of the Pythagorean Theorem. Make sure to brush up on geometry and trigonometry.

The magnitude of a vector can be computed with the Pythagorean Theorem. The magnitude of vector **A** can be denoted in several ways: A or $|\mathbf{A}|$ or $\|\mathbf{A}\|$. In terms of its components, the magnitude of $\mathbf{A} = A_x\mathbf{\hat{i}} + A_y\mathbf{\hat{j}}$ is given by the equation

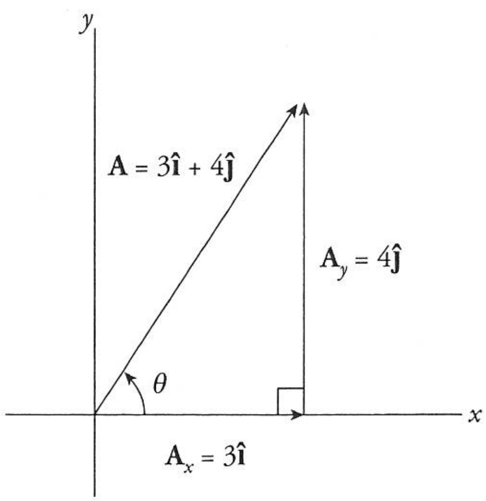
$$A = \sqrt{(A_x)^2 + (A_y)^2}$$

which is the formula for the length of the hypotenuse of a right triangle with sides of lengths A_x and A_y.



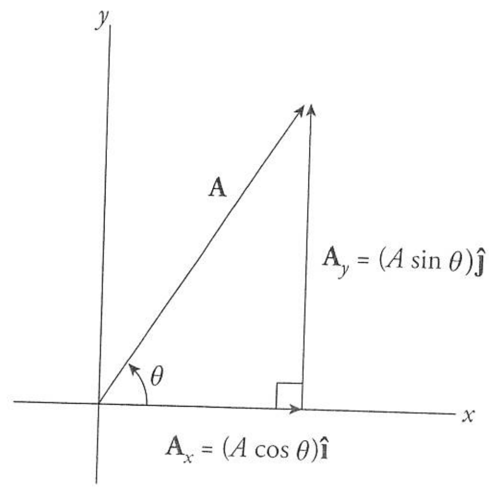
DIRECTION OF A VECTOR

The direction of a vector can be specified by the angle it makes with the positive x-axis. You can sketch the vector and use its components (and an inverse trig function) to determine the angle. For example, if θ denotes the angle that the vector $A = 3\hat{\imath} + 4\hat{\jmath}$ makes with the $+x$-axis, then $\tan\theta = 4/3$, so $\theta = \tan^{-1}(4/3) = 53.1°$.



In general, the axis that θ is made to is known as the adjacent axis. The adjacent component is always going to get the cos θ. For example, if **A** makes the angle θ with the +x-axis, then its x- and y-components are $A \cos \theta$ and $A \sin \theta$, respectively (where A is the magnitude of **A**).

$$\mathbf{A} = \underbrace{(A\cos\theta)}_{A_x}\hat{\mathbf{i}} + \underbrace{(A\sin\theta)}_{A_y}\hat{\mathbf{j}}$$



In general, any vector in the plane can be written in terms of two perpendicular component vectors. For example, vector **W** (shown below) is the sum of two component vectors whose magnitudes are $W \cos \theta$ and $W \sin \theta$: