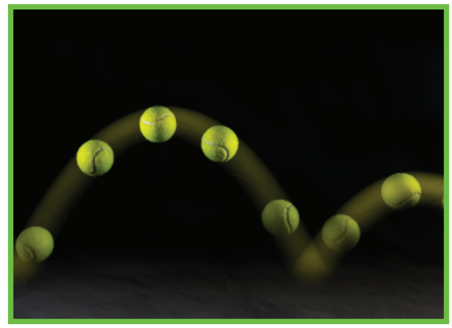
4-5 Geometric Sequences

Objective To recognize and extend geometric sequences • To find an indicated term of a geometric sequence • To write a recursive formula for a geometric sequence

Some students are measuring the progress of a bouncing ball. In one experiment, the ball is dropped from a height of 16 feet to the floor below. After the first bounce, the ball rebounds to a height of 4 feet above the floor; after the second bounce, it rebounds to a height of 1 foot; and after the third bounce, it rebounds to a height of $\frac{1}{4}$ foot. If the ball continues to bounce and rebound at this rate, what will be the height of the ball after the fourth bounce?



To find the height after the fourth bounce, look for a pattern that occurs during the first three bounces.

► To look for a pattern, organize the data by using a table or by using a graph.

- Use a Table

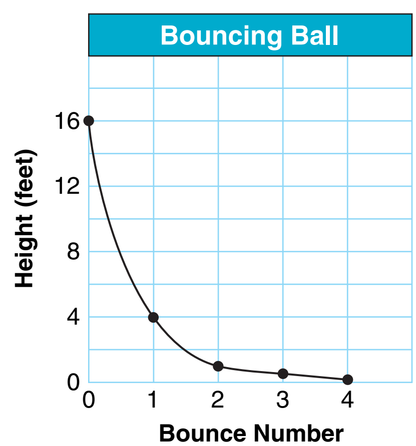
Bounce Number	0	1	2	3	4
Height (feet)	16	4	1	$\frac{1}{4}$	?

$\div 4$ $\div 4$ $\div 4$ ← Divide by 4 $\left(\text{or multiply by } \frac{1}{4}\right)$ to get the next number.

According to the pattern, you can find the height of the ball after the fourth bounce by dividing its height after the third bounce by 4. So after the fourth bounce, the height of the ball is $\frac{1}{4} \div 4 = \frac{1}{4} \bullet \frac{1}{4} = \frac{1}{16}$ foot.

- Use a Graph

When the information about the bouncing ball is shown on a graph, the points appear to fall on a curve.



Applying the vertical-line test to this graph shows that this relation is a function.

You can find the height of the ball after the fifth bounce by continuing the pattern: $16, 4, 1, \frac{1}{4}, \frac{1}{16}, \frac{1}{16} \div 4 = \frac{1}{16} \bullet \frac{1}{4} = \frac{1}{64}, \ldots$

► A sequence in which each term after the first is found by *multiplying* the previous term by a constant (called the common ratio, r) is called a geometric sequence.

In the sequence, $16, 4, 1, \frac{1}{4}, \frac{1}{16}, \frac{1}{64}$:

$4 = 16 \bullet \frac{1}{4}$ $\qquad \frac{1}{4} = 1 \bullet \frac{1}{4}$ $\qquad \frac{1}{64} = \frac{1}{16} \bullet \frac{1}{4}$

$1 = 4 \bullet \frac{1}{4}$ $\qquad \frac{1}{16} = \frac{1}{4} \bullet \frac{1}{4}$

Note that the constant $\frac{1}{4}$ may be found by dividing any term by its preceding term.

$\frac{4}{16} = \frac{1}{4}$ $\qquad \frac{\frac{1}{4}}{1} = \frac{1}{4}$ $\qquad \frac{\frac{1}{16}}{\frac{1}{4}} = \frac{1}{\cancel{16}_{4}} \bullet \frac{\cancel{4}^{1}}{1} = \frac{1}{4}$ $\qquad \frac{\frac{1}{64}}{\frac{1}{16}} = \frac{1}{\cancel{64}_{4}} \bullet -\frac{\cancel{16}^{1}}{1} = \frac{1}{4}$

Examples

Determine if each sequence is a geometric sequence. If the sequence is geometric, find the next term.

Think Test for a common ratio.

1 $-\frac{1}{4}, \frac{1}{2}, -1, 2, \ldots$

$\frac{a_2}{a_1} = \frac{\frac{1}{2}}{-\frac{1}{4}} = \frac{1}{2}\left(-\frac{4}{1}\right) = -2$ | $\frac{a_3}{a_2} = \frac{-1}{\frac{1}{2}} = -\frac{1}{1} \bullet \frac{2}{1} = -2$ | $\frac{a_4}{a_3} = \frac{2}{-1} = -2$ ← There is a common ratio, $r = -2$.

So the sequence $-\frac{1}{4}, \frac{1}{2}, -1, 2, \ldots$ is a geometric sequence.

The next term is $2(-2) = -4$.

2 $10, 3\frac{1}{3}, 1\frac{1}{9}, \frac{10}{27}, \ldots$

$10, \frac{10}{3}, \frac{10}{9}, \frac{10}{27}, \ldots$ ← Rewrite mixed numbers as improper fractions.

$\frac{a_2}{a_1} = \frac{\frac{10}{3}}{10} = \frac{10}{3} \bullet \frac{1}{10} = \frac{1}{3}$ | $\frac{a_3}{a_2} = \frac{\frac{10}{9}}{\frac{10}{3}} = \frac{10}{9} \bullet \frac{3}{10} = \frac{1}{3}$ | $\frac{a_4}{a_3} = \frac{\frac{10}{27}}{\frac{10}{9}} = \frac{10}{27} \bullet \frac{9}{10} = \frac{1}{3}$ ← There is a common ratio, $r = \frac{1}{3}$.

So $10, 3\frac{1}{3}, 1\frac{1}{9}, \frac{10}{27}, \ldots$ is a geometric sequence.

The next term is $\frac{10}{27}\left(\frac{1}{3}\right) = \frac{10}{81}$.

► Each term of a geometric sequence can be related to the first term, a_1, and the common ratio, r. The sequence may be written as follows:

Number of Term	1	2	3	4	5	...	a_n
	↕	↕	↕	↕	↕		↕
Term of Sequence	a_1	a_1r	a_1r^2	a_1r^3	a_1r^4	...	a_1r^{n-1}

In general, the nth term of a geometric sequence is the product of the first term, a_1, and the common ratio, r, raised to the $(n - 1)$ power, that is, $a_n = a_1r^{n-1}$.

Key Concept

Formula for the nth term of a Geometric Sequence

$a_n = a_1r^{n-1}$, where a_n is the nth term, a_1 is the first term, n is the number of terms, and $r (r \neq 0, 1)$ is the common ratio.

Find a_9 of 2, 20. 200, 2000, . . .

- Find r by dividing the second term by the first term.

$r = \frac{a_2}{a_1} = \frac{20}{2} = 10$

- Use the formula to find the nth term of a geometric sequence.

$a_n = a_1r^{n-1}$

$a_9 = 2 \bullet 10^{9-1}$ ← Substitute 9 for n, 2 for a_1, and 10 for r.

$= 2 \bullet 10^8$

$= 2 \bullet 100{,}000{,}000$ ← Simplify.

$= 200{,}000{,}000$

Continue Lesson ➡

Examples

1 What is the 10th term of the geometric sequence given that the first term is −128, and the common ratio is 0.5?

$a_n = a_1 \quad r^{n-1}$ ← Write the rule to find the *n*th term.

$a_{10} = -128 \quad (0.5)^{10-1}$ ← Substitute 10 for *n*, −128 for a_1, and 0.5 for *r*.

$a_{10} = -0.25$ ← Simplify.

2 Find a_7 of $-8, 4, -2, 1, \ldots$.

$r = \frac{a_2}{a_1} = \frac{4}{-8} = -\frac{1}{2}$ ← Find *r* by dividing the second term by the first term.

$a_7 = -8 \bullet \left(-\frac{1}{2}\right)^{7-1}$ ← Substitute 7 for *n*, −8 for a_1, and $-\frac{1}{2}$ for *r* in $a_n = a_1 r^{n-1}$.

$= -8 \bullet \left(-\frac{1}{2}\right)^{6} = -8 \bullet \frac{1}{64}$ ← Simplify.

$= -\frac{8}{64} = -\frac{1}{8}$

► You can also use a recursive formula to find the *n*th term, a_n, of a geometric sequence by using the preceding term, a_{n-1}, and the common ratio, *r*. A recursive formula is a rule for calculating a new term of a sequence from the term preceding it.

In general, in a geometric sequence, if *r* represents the common ratio and a_{n-1} and a_n represent two consecutive terms, then $r = \frac{a_n}{a_{n-1}}$. This will lead to the recursive formula for the *n*th term, a_n.

$r \bullet a_{n-1} = \frac{a_n}{a_{n-1}} \bullet a_{n-1}$ ← Use the Multiplication Property of Equality.

$= \frac{a_n}{\cancel{a_{n-1}}_1} \bullet \cancel{a_{n-1}}^1$ ← Divide out common factors.

$= a_n$ ← Simplify.

Key Concept

Recursive Formula for the *n*th Term of a Geometric Sequence

$a_n = ra_{n-1}$, where a_n is the *n*th term, *n* is the number of terms, and $r (r \neq 0, 1)$ is the common ratio.

Write a recursive formula for the *n*th term of the sequence: $8, -28, 98, -343, \ldots$

$r = \frac{a_n}{a_1} = \frac{-28}{8} = -\frac{7}{2}$ ← Find *r*, by dividing the second term by the first term.

$a_n = -\frac{7}{2}a_{n-1}$ ← Substitute −3 for *r* in $a_n = ra_{n-1}$.

Example

1 Write a recursive formula for the *n*th term of the geometric sequence:

$\frac{1}{4}, -\frac{3}{4}, \frac{9}{4}, -\frac{27}{4}, \ldots$

$r = \frac{a_2}{a_1} = \frac{-\frac{3}{4}}{\frac{1}{4}} = \frac{-3}{\cancel{4}_1} \bullet \frac{\cancel{4}^1}{1} = -3$ ← Find *r*, by dividing the second term by the first term.

$a_n = -3a_{n-1}$ ← Substitute −3 for *r* in $a_n = ra_{n-1}$.

Try These

Determine whether each sequence could be *geometric*, *arithmetic*, or *neither*. Then find the next four terms using a pattern.

1. 456, 45.6, 4.56, 0.456, . . .

2. $3^3, 3^{-3}, 3^3, 3^{-3}, \ldots$

3. 0.07, 0.08, 0.09, 0.10, . . .

4. $\frac{3}{4}, \frac{1}{2}, \frac{1}{3}, \frac{2}{9}, \ldots$

5. $\frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \ldots$

6. $\frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots$

Find the indicated term of each geometric sequence.

7. a_8 for $1, -5, 25, -125, \ldots$

8. a_7 for 3, 0.6, 0.12, 0.024, . . .

9. a_9 for $x, 2x, 4x, 8x, \ldots$

Write a recursive formula for the *n*th term of each geometric sequence.

10. 0.3, 1.2, 4.8, 19.2, . . .

11. $-\frac{2}{81}, \frac{2}{27}, -\frac{2}{9}, \frac{2}{3}, \ldots$

12. $y^{-4}, y^{-2}, 1, y^2, \ldots$

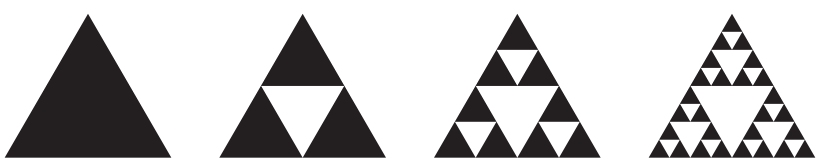
13. Hilda bought a car for $17,500. Her car depreciated 25% in value each year. What was Hilda's car worth 3 years after she bought it?

14. Discuss and Write Barry has written a geometric sequence. In his sequence, the third term is $\frac{3}{2}$, and the seventh term is $\frac{3}{32}$. What is the first term of Barry's sequence? Explain your reasoning.

Go to PRACTICE BOOK Lesson 4-5 for exercise sets.

Acceleration: Geometric Sequences of Fractals

A fractal is a fragmented geometric shape that is subdivided in parts, each of which is similar to the entire shape. Waclaw Sierpinksi, a Polish mathematician, created fractals using a triangle. The figures below show the first four steps in making Sierpinksi's Triangle.



Step 1 Step 2 Step 3 Step 4

To create each step, connect the midpoints of the sides of the shaded triangles at the previous step and remove the middle triangles.

Look at the number of shaded triangles at each stage. What fraction of each step is shaded?

	Step 1	Step 2	Step 3	Step 4
Number of shaded triangles →	1	3	9	27
Fraction shaded →	1	$\frac{3}{4}$	$\frac{9}{16}$	$\frac{27}{64}$

Notice that $1, \frac{3}{4}, \frac{9}{16}, \frac{27}{64}$ is a geometric sequence with a common ratio, $\frac{3}{4}$.

Write a recursive formula for the *n*th term of the geometric sequence. $a_n = \frac{3}{4}a_{n-1}$

What fraction of Step 5 is shaded? $\frac{81}{256}$

Explain how you arrived at your answer. $a_5 = \frac{3}{4} \bullet \frac{27}{64} = \frac{81}{256}$